Semiquantal corrections to stochastic resonance

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Stochastic resonance occurs when a nonlinear dynamical system is periodically modulated in the presence of noise. We study in this paper the effect of including quantal corrections on the thermal transition rates of such a system, by using the Gaussian effective potential. We demonstrate that these corrections sharpen the peak of the signal-to-noise ratio and shift it toward a lower noise variance, in general. By studying the symmetric double-well system over a range of parameters, we are also able to show that the extreme quantal case of this system demonstrates no stochastic resonance, in agreement with recent calculations.

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I. INTRODUCTION

Nonlinear dynamical systems when periodically modulated in the presence of noise exhibit the phenomenon of stochastic resonance [1–8]. This is manifested as a maximum in the signal-to-noise ratio (SNR) as a function of noise strength. The canonical example of such a system is a one-dimensional bistable system (a double-well potential; see Fig. 1) that is subject both to random fluctuations (characterized equivalently by a temperature or the variance of these fluctuations) and a small-amplitude sinusoidal modulation of the potential characterized by its frequency. The response, i.e., the "signal" is then the transition between the two wells where for simplicity we consider a symmetric potential. In the absence of the fluctuations, there are no transitions of the system between the two wells unless the driving amplitude is so large as to effectively eliminate the central barrier, in which case the modulation of the system and the response are fully correlated. In the small-amplitude regime (where we stay throughout this paper), there is thus no signal in the absence of fluctuations. Adding some noise (with small variance) creates a nonzero probability of transition. As the variance of the fluctuations is increased, the probability of transition is also increased. Given that the probability of transition depends on the barrier height, we hence expect there to be increasing correlation between the sinusoidal modulation of the barrier height and the system response as the noise variance is increased from zero. At arbitrarily large noise variances, however, we expect the noise to render the barrier (and its small-amplitude variation) effectively invisible; there should thus be no correlation between the driving and the system response. This intuition is quantified in a peak in a graph of SNR versus noise variance, and is what is termed stochastic resonance [5]. A variety of systems has been argued to exhibit this phenomenon; this has been verified by experiments and simulations in systems ranging from the macroscopic (the Earth's climate [3]) through the mesoscopic (e.g., the bistable ring laser [4]) to the microscopic (e.g., the Schmitt trigger [6] and SQUID loops [7]).

Although these systems have included those that are classified as "quantum" systems, the usual analysis of stochastic resonance has been from the perspective of a classical potential system. We now address the question of the inclusion of quantum effects in considering the phenomenon of stochastic resonance.

Recently, Löfstedt and Coppersmith [8] have analyzed the notion of quantum stochastic resonance. Their model of the two-well system is a quantum-mechanical two-state system coupled to an Ohmic heat-bath from which the dissipation is obtained. The signal is then the transition between the two states; the transition rates are obtained [9] with the canonical Caldeira-Leggett approach [10]. Among the other results they have obtained is the demonstration that there is no quantum stochastic resonance for the symmetric double-well problem. Their approach lies in the extreme quantum regime; patently,

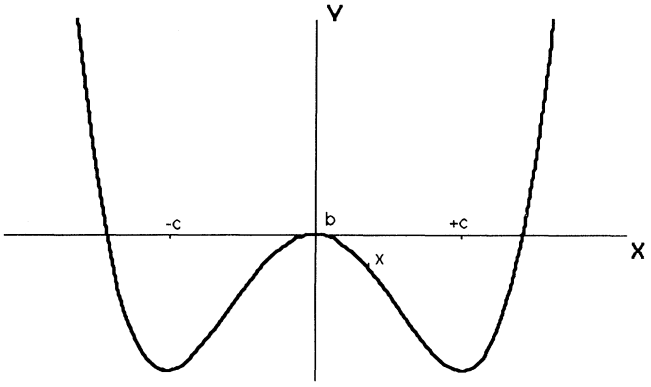

FIG. 1. The double-well potential $-\gamma x^2/2+\beta x^4/4$ in arbitrary units; γ and β are arbitrary here. $x=b$ and $x=c$ are the maximum and one of the minima, respectively.

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since quantum mechanics underlies classical mechanics, and the classical symmetric problem does display stochastic resonance (it is in fact the paradigm for the phenomenon) this result cannot hold true in the classical limit of the symmetric double-well quantum system. The seeming contradiction is easily resolved by the realization that the classical limit of the double-well system is not a two-state system but involves, in fact, all the low-lying states of the potential. In this limit, an analysis similar to Löfstedt and Coppersmith would then be very complicated since a multitude of transition rates would have to be considered, and the notion of system transitions becomes hard to pin down. Physical intuition and the correspondence limit, however, indicate that there should be a way to understand the near-classical problem as the classical problem with "corrections." In this paper, we provide a way of doing so using the semiquantal approach of the Gaussian effective potential (GEP) [11,12].

The GEP provides a way of quantifying our intuition of the qualitative changes expected in a dynamical problem if quantum effects are considered. For the double-well problem, the basic effects expected are as follows. First, we expect the effect of quantal tunneling processes to effectively lower the height of the central barrier. Secondly, we expect the zero-point quantum fluctuation to effectively raise the minimum of each potential well. A third effect, which has a minor impact on the end result but complicates the calculations, is that the presence of states symmetric about the central barrier leads to an effective minimum at the barrier itself. This is a shallow minimum in the classical limit, as we shall demonstrate. Accounting for its presence, however, ultimately helps us recover the conclusions of Löfstedt and Coppersmith in the extreme quantum regime.

The paper is organized as follows. In the next section, we motivate the GEP, starting from Ehenfest's theorem, and demonstrate that its derivation remains valid in the presence of driving and a classical noise source. We then derive an explicit expression for the noise-driven transition rates for the semiquantal system to which end we adapt the theory of McNamara and Weisenfeld [5] on the classical problem. In Sec. IV, we present the quantitative results of including quantum effects on the classical stochastic resonance problem. We conclude with a short discussion.

II. THE GAUSSIAN EFFECTIVE POTENTIAL (GEP)

The GEP has been used to study quantum effects such as zero-point fluctuations and tunneling on the geometry of classical potentials in field theory and ordinary quantum mechanics [11]. A recent derivation of the GEP from Ehrenfest's theorem [12] has enabled its use in dynamical problems as well. This has been used to demonstrate that quantum effects destroy chaos in the classically chaotic Hénon-Heiles problem, dominantly, through tunneling. We are thus motivated to extend the use of the GEP to understand the impact of quantum effects on the phenomenon of classical stochastic resonance as well.

In this section we rederive the GEP, extending our analysis to the case of a time-dependent potential in the presence of a classical noise source. Before we proceed to the derivation of the dynamical equations, however, we specify our model system. We assume a Hamiltonian of the form

$$\hat{H}=\frac{\hat{p}^2}{2}+V_0(\hat{x})+\hat{x}F(t)\ , \tag{1}$$

where $V_0(\hat{x})$ specifies the potential of the "unperturbed" problem. We use linear coupling to introduce the external environment; the term $F(t)$ includes both the small amplitude perturbation and the effect of a *classical* random noise source. The starting point is the equations of motion for the centroid of a wave packet:

$$\frac{d}{dt}\langle\hat{x}\rangle=\langle\hat{p}\rangle\ , \tag{2}$$

$$\frac{d}{dt}\langle\hat{p}\rangle=-\left\langle\frac{\partial V_0(\hat{x})}{\partial x}\right\rangle-F(t)\ , \tag{3}$$

where $\langle\ \rangle$ denotes the quantum expectation value. In general, these two ordinary differential equations (ignoring for the moment the stochastic variable) camouflage an infinite system of ordinary differential equations through the nonlocal character of the function $\langle\partial V_0(\hat{x})/\partial x\rangle$. We may, however, approximate the system by a finite one by first making the Taylor series expansions of all functions $G(\hat{u})$ as

$$\langle G(\hat{u})\rangle=\frac{1}{n!}\langle\hat{\Delta}u^n\rangle G^{(n)}\ , \tag{4}$$

where $\hat{\Delta}u^n=\hat{u}-\langle\hat{u}\rangle$ and $G^{(n)}=\partial^n G/\partial u^n|_{\langle\hat{u}\rangle}$. We can then either truncate the infinite system at a given finite order or take the semiquantal approach in making the time-dependent variational ansatz that the wave packet is always of the (normalized) form

$$\Psi(q,t)=(2\pi\rho^2)^{-1/4}\exp\{i[A(q-x)^2+p(q-x)]\}\ . \tag{5}$$

This implies the relations

$$\langle\hat{x}\rangle=x\ , \tag{6}$$

$$\langle\hat{p}\rangle=p\ , \tag{7}$$

$$\langle\Delta x^{2m}\rangle=\frac{(2m)!\rho^{2m}}{m!2^m}\quad\text{(no summation)}\ , \tag{8}$$

$$\langle\Delta x^{2m+1}\rangle=0\ , \tag{9}$$

$$\rho^2\langle\Delta p^2\rangle=\frac{\hbar^2}{4}+\rho^2\Pi^2\ , \tag{10}$$

$$\langle\Delta x\Delta p+\Delta p\Delta x\rangle=2\rho\Pi\ , \tag{11}$$

where

$$A=\frac{\Pi}{2\rho}+i\frac{1}{4\rho^2}\ . \tag{12}$$

The variables ρ,Π are thus the quantal "fluctuation" variables.

This yields the following equations of motion of the wave packet in terms of its parameters:

$$\frac{dx}{dt}=p\ , \tag{13}$$

$$\frac{dp}{dt}=-\sum_m \frac{\rho^{2m}}{m!2^m}V_0^{(2m+1)}(x)-F(t)\ ,$$

$$m=0,1,\ldots, \quad (14)$$

$$\frac{d\rho}{dt}=\Pi\ , \quad (15)$$

$$\frac{d\Pi}{dt}=\frac{\hbar^2}{4\rho^3}-\sum_m \frac{\rho^{2m-1}}{(m-1)!2^{m-1}}V_0^{(2m)}(x)\ ,$$

$$m=1,2,\ldots . \quad (16)$$

As in the absence of the external force, this yields a classical *extended potential system* as our approximation to the Hilbert space. The classical degrees of freedom are now the "average" variables x,p and the "quantal" variables ρ,Π, respectively; the associated Hamiltonian is

$$H_{\rm ext}=\frac{p^2}{2}+\frac{\Pi^2}{2}+V_{\rm ext}(x,\rho)\ , \quad (17)$$

$$V_{\rm ext}(x,\rho)=V_0(x)+xF(t)+\frac{\hbar^2}{8\rho^2}+\sum_m \frac{\rho^{2m}}{m!2^m}V_0^{(2m)}(x)\ ,\quad m=1,2\ldots , \quad (18)$$

where the subscript ext indicates the "extended" potential and Hamiltonian.

The structure and implications of this extended Hamiltonian formulation have been previously discussed in detail [13]. The crucial point to note in this derivation is that since the noise and modulation were introduced through linear coupling to the original problem, they appear only in the equations for the average or "classical" variables x,p. Hence, all previous considerations of this system carry through, taking care only to introduce the external force where appropriate. In particular, the adiabatic elimination of the "quantal" variables ρ,Π proceeds as before. This is done under the usual "slaving" assumption [14] that these variables have a dynamical time scale much shorter than that of the average variables.

With this assumption, we solve for the steady-state values of the variables ρ,Π from Eqs. (15) and (16) and substitute these (as functions of x) in Eq. (14). In the extended potential formulation this has the even simpler mathematical meaning of minimizing the Hamiltonian $H_{\rm ext}$ with respect to these variables to yield $\rho_{\rm min}(x),\Pi_{\rm min}(x)$. Patently, for the particular form of the Hamiltonian chosen above, $\Pi_{\rm min}\equiv 0$. We need thus solve only for $\rho_{\rm min}(x)$; substituting this back in $V_{\rm ext}$ yields a function $V_{\rm eff}(x)$ which is termed the Gaussian effective potential (GEP).

We now apply this procedure to the double-well Hamiltonian. With

$$\hat{H}=\frac{\hat{p}^2}{2}-\frac{\gamma\hat{x}^2}{2}+\frac{\beta\hat{x}^4}{4}\ , \quad (19)$$

the extended Hamiltonian is

$$H_{\rm ext}=\frac{p^2}{2}+\frac{\Pi^2}{2}-\frac{\gamma}{2}(x^2+\rho^2)+\frac{1}{8\rho^2}+\frac{\beta}{4}(x^4+3\rho^4+6x^2\rho^2)\ . \quad (20)$$

Minimizing as above leads to the effective Hamiltonian

$$H_{\rm eff}=\frac{p^2}{2}+V_{\rm eff}(x)\ , \quad (21)$$

$$V_{\rm eff}=\frac{\gamma x^2}{2}+\frac{\beta x^4}{4}+\frac{\hbar\Omega}{2}-\frac{3\hbar^2\beta}{16\Omega^2}\ , \quad (22)$$

where we define

$$\Omega\equiv\frac{1}{\rho^2}\ . \quad (23)$$

Ω is the largest solution of the equation

$$\Omega^3+(\gamma-3\beta x^2)\Omega-\frac{3\hbar\beta}{2}=0 \quad (24)$$

and is hence an implicit function of x.

Equations (21)–(24) define the effective potential for our problem consisting of the classical potential with semiclassical or semiquantal corrections. We now study its shape for various parameter ranges.

There are three parameters to be considered: $\hbar,\gamma,\beta$. The system geometry is, however, completely defined by the single scaling parameter ξ defined as

$$\xi^3\equiv\frac{32\gamma^3}{243\hbar^2\beta^2}\ , \quad (25)$$

which is dimensionless in units of $\hbar$. This parameter thus measures the quantal nature of the system, with $\xi\to 0$ defining the extreme quantal case and $\xi\to\infty$ the extreme classical case. A reference scale may be set by considering the ammonia molecule. The double-well potential may now be seen as the bistable system in which the nitrogen atom resides, with the barrier being the plane of the hydrogen triad. With a mass 5×10^{-26} kg scaled to unity, with a well depth of 4×10^{-20} J, and minima separation of 4×10^{-11} m, the parameters may be scaled as $\gamma=1,\beta=0.004,\hbar=0.05$, which scales the time to be in units of 10^{-14} sec. This corresponds to $\xi=148.8$. The appropriate semiquantal potential is shown in Fig. 2, overlaid on the classical potential ($\hbar\equiv 0$) for the same parameters. The dominant effect is the diminishing of the central barrier height by a factor of about $\frac{1}{3}$ through quantal tunneling effects.

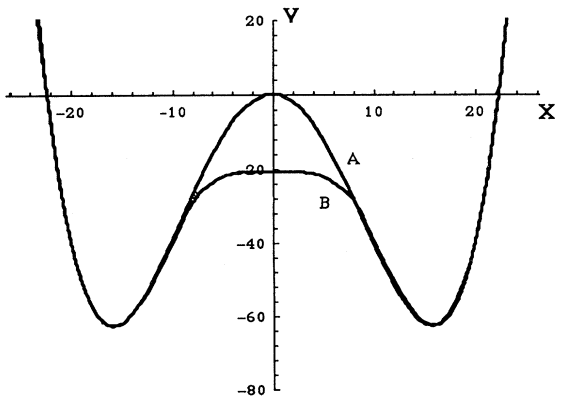

FIG. 2. Potentials with $\xi=148.8$ ($\gamma=1.0$, $\beta=0.004$, $\hbar=0.05$). *A*, classical potential with $\hbar=0$; *B*, semiquantal potential.

As ξ is lowered, we expect to see the other quantal effects discussed in the Introduction. Figure 3 shows the semiquantal potential overlaid on the appropriate classical potential at $\xi=2.0$. We now see the minima in the two wells elevated with respect to the classical minima due to zero-point fluctuation effects. This appears at all scales, of course, but is more visible proportional to the diminishing well depth at this case. For this range of ξ the minima are elevated by a factor of about $\frac{1}{3}$ and the barrier lowered again by a factor of $\frac{1}{3}$, leading to a semiquantal well of approximately $\frac{1}{3}$ the classical well. Lowering ξ even further brings out yet another quantum effect. At $\xi=1.2$, shown in Fig. 4, we see a minimum appearing at $x=0$. This is due to the lower energy of the symmetric states compared to states displaced from the symmetry axis. The extreme case of this effect is shown in Fig. 5 with $\xi=1.0$ where the quantal fluctuations and tunneling have rendered the central barrier invisible and the system effectively resides in a single well. At this scale, there can be no transitions and hence no semiquantal stochastic resonance. This result in the extreme quantal case thus agrees with the result of Löfstedt and Coppersmith, who have argued that there is no quantum stochastic resonance for the symmetric system. We point out, however, that the inherent difference in our two approaches is that the transition rates with which they look for stochastic resonance are the quantum-mechanical tunneling rates, while we consider the thermal transition rates (as in the classical system) and are estimating the effect of tunneling and quantal fluctuations on these rates.

We calculate these effects for the case of larger ξ down to $\xi=1.2$ approximately, in the next section.

III. CALCULATION OF TRANSITION RATES

The calculation of transition rates for classical stochastic resonance in the double-well problem is done most simply by following McNamara and Weisenfeld [5]. In this method, one starts with a two-state master equation; under the usual assumption that the drive frequency and interwell transition rates are much slower than the intrawell relaxation rate, the time-dependent transition rates can then be derived. This enables the computation of an autocorrelation function, and thus the signal-to-noise ratio (SNR). Recent nonperturbative calculations show that the result thus perturbatively derived provides a good qualitative description of the dynamics even when the driving modulation of the potential is not small [8].

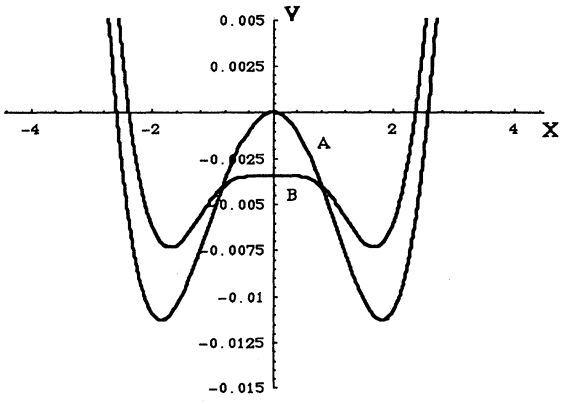

FIG. 3. Same as Fig. 2, $\xi=2.0$.

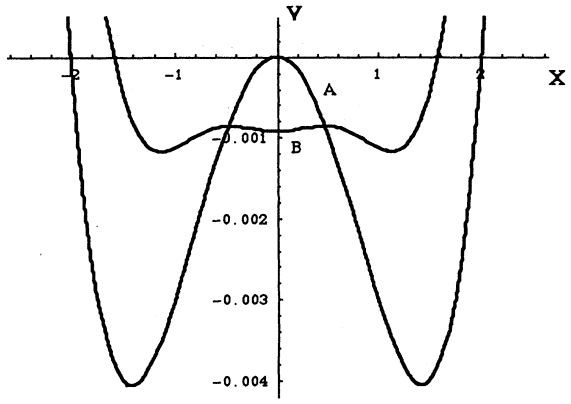

FIG. 4. Same as Fig. 2, $\xi=1.2$.

To use this method, we specify $F(t)$ as

$$F(t)=\epsilon\cos(\omega t)+\zeta(t)\ , \tag{26}$$

where ϵ is the small amplitude of the modulation and ζ is a *classical* noise variable which is assumed to have the properties

$$\bar{\zeta}=0\ , \tag{27}$$

$$\overline{\zeta(t_1)\zeta(t_2)}=2D\delta(t_1-t_2)\ , \tag{28}$$

where the overbar represents the stochastic average. D is hence the variance of the noise and a measure of its intensity. Under these conditions, therefore, the SNR of stochastic resonance for a double-well system (independent of its precise shape) is computed to be [5]

$$\mathcal{R}_{\rm SNR}\approx\frac{\pi}{4}\frac{\alpha_1^2\eta^2}{\alpha_0}\ . \tag{29}$$

The symbols multiplying the constant $\pi/4$ on the right-hand side have the following meanings: η is ϵ scaled by

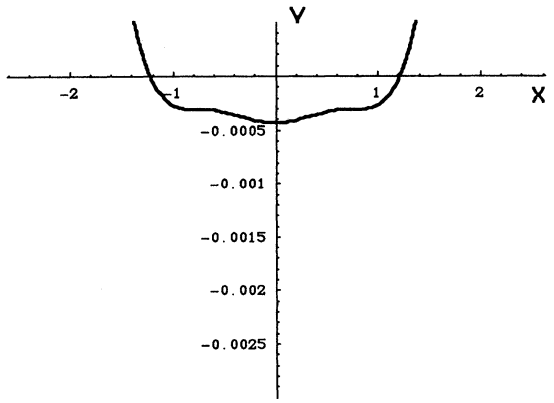

FIG. 5. Same as Fig. 2, $\xi=1.0$. Note that the system is effectively a single well.

the variance D of the noise and the geometry of the potential as detailed below. α_0 and α_1 are defined by the equations

$$W_{\pm}(t)=f(\mu\pm\eta\cos(\omega t)) \tag{30}$$

$$=\tfrac{1}{2}[\alpha_0\mp\alpha_1\eta\cos(\omega t)+\alpha_2\eta^2\cos^2(\omega t)\mp\cdots], \tag{31}$$

where the $W_{\pm}$ are the transition rates from the $\pm$ states, defined as the left- and right-hand well, respectively. These have been assumed to be some function f which can be expanded around its value at $\eta=0$ (equivalently around μ) as shown, where μ is the static potential barrier height δV also scaled by the noise variance D. The expansion coefficients α_n in the perturbative calculation of the time-dependent transition rates can be computed as

$$\tfrac{1}{2}\alpha_n=\frac{(-1)^n}{n!}\frac{d^n f}{d[\eta\cos(\omega t)]}\bigg|_{\eta=0}. \tag{32}$$

To obtain these quantities, therefore, we need to calculate $W_{\pm}$ for our semiquantal system. This is usually done using Kramer's rule [15] to give the escape time out of either well as

$$W_K^{-1}=\tau_K=\frac{2\pi}{\sqrt{|V^{(2)}_{\max}||V^{(2)}_{\min}|}}\exp(V_{\max}-V_{\min}), \tag{33}$$

where $V_{\max}$ and $V_{\min}$ are the values of the potential at the central barrier and at either minimum, the superscript (2) denotes the second derivative of the potential at that point, and the vertical bars denote taking the absolute value. Note that the semiquantal approach requires the replacement of $V(x)$ by $V_{\text{eff}}(x)$. However, the V_{eff} we have derived in the preceding section does not permit the use of this formula in its entirety. This is because the shape of the potential obtained creates a shallow minimum at the classical $V_{\max}$. While this does not affect our treatment of the system as an essentially double-well problem (at least up to $\xi=1.2$), it does invalidate the parabolic approximation used in deriving Kramer's rule: we would get artificially large escape times from this formula. To obtain the correct form, we have to return to the derivation of Kramer's rule to find our point of departure.

The solution for the escape time out of a potential well with one absorbing and one reflecting barrier, using the backward Fokker-Planck equation, can be shown to be [15]

$$T(c\to x_0)=\frac{1}{D}\int_c^{x_0}dy\exp\left[\frac{V(y)}{D}\right]\times\int_{-\infty}^{y}dz\exp\left[\frac{-V(z)}{D}\right], \tag{34}$$

where c is, as labeled in Fig. 1, the initial position (at one of the minima) and x_0 is in the vicinity of, but beyond the maximum at, b. D is the variance of the fluctuation variable as before. The usual approximation of this double integral by a product of integrals holds even for our system. The argument goes as follows. Near the central maximum ($x\simeq 0$), $V(x)$ has its greatest value. Therefore, given that D is small, $\exp(-V(z)/D)$ is very small around $z=0$. Hence, the integral over z of $\exp(-V(z)/D)$ is a very slowly varying function of y (its upper limit) near $y=0$. On the other hand, $\exp(V(y)/D)$ has a sharply peaked *maximum* around $y=0$. This enables us to replace the upper limit of the integral over z with 0, thus making it independent of the y integral. We now have the product of two single integrals:

$$T(c\to x_0)\simeq\frac{1}{D}\left[\int_{-\infty}^{0}dz\exp\left[\frac{-V(z)}{D}\right]\right]\times\left[\int_c^{x_0}dy\exp\left[\frac{V(y)}{D}\right]\right]. \tag{35}$$

The exponentially dominant contributions to the two integrals are around the extrema and the above expression can be very well approximated by

$$T(c\to x_0)\simeq\frac{1}{D}\left[\int_{c-\delta_c}^{c+\delta_c}dz\exp\left[\frac{-V(z)}{D}\right]\right]\times\left[\int_{b-\delta_b}^{b+\delta_b}dy\exp\left[\frac{V(y)}{D}\right]\right], \tag{36}$$

where the neighborhoods $-\delta$ to $+\delta$ are chosen around the extrema such that the value of the integrals are insensitive to the precise choice of δ.

At this point the two integrals are usually evaluated [15] through the parabolic approximation where the integrand is expanded around the extrema and leads to Kramer's rule. In our case, this approximation can only be applied to the integral around the minimum, where it is clearly valid. We then get

$$\int_{c-\delta_c}^{c+\delta_c}dz\exp\left[\frac{-V(z)}{D}\right]\simeq\int_{-\infty}^{\infty}dz\exp\left[\frac{-V(c)}{D}-\frac{(z-c)^2}{2D}V^{(2)}(c)\right] \tag{37}$$

$$=\left[\frac{2\pi D}{V^{(2)}(c)}\right]^{1/2}\exp\left[\frac{-V(c)}{D}\right]. \tag{38}$$

The maximum is too slowly varying for the parabolic approximation; hence, our final expression for the semiquantal transition rate T_{sq} is

$$T_{\rm sq}(c\to x_0)=\frac{1}{D}\left[\frac{2\pi D}{V''_{\rm eff}(c)}\right]^{1/2}\exp\left[\frac{-V_{\rm eff}(c)}{D}\right]\left[\int_{b-\delta_b}^{b+\delta_b}dy\exp\left[\frac{V_{\rm eff}(y)}{D}\right]\right]_{\rm num} \tag{39}$$

with the subscript num indicating a factor to be computed numerically; note that we have now explicitly indicated that the potential we are using is the semiquantal one $V_{\rm eff}$. This expression can now be used for the comparison of the semiquantal system with the associated classical one.

The final step is to use this expression to compute the quantities α_0 and α_1. We shall do this with the same perturbative approach as McNamara and Wiesenfeld [5]. With the potential modulation, the above expression transforms to

$$T_{\rm sq}^{\rm drive}=\frac{1}{D}\left[\frac{2\pi D}{V''_{\rm eff}(c)}\right]^{1/2}\exp\left[\frac{-V_{\rm eff}(c)}{D}+\frac{\epsilon c}{D}\cos(\omega t)\right]\left[\int_{-\delta}^{+\delta}dy\exp\left[\frac{V_{\rm eff}(y)}{D}+\frac{\epsilon y}{D}\cos(\omega t)\right]\right]_{\rm num}, \tag{40}$$

where we have now explicitly placed the barrier at $x=0$. In including the modulation in the first term, we approximate the minimum as remaining at $\pm c$. We have to invert this expression to find the transition rate W. We can then compute α_0 and α_1. Noticing that $\epsilon c/D$ is what we have called η earlier, we get

$$\alpha_0=2f(\eta=0) \tag{41}$$

$$=\left[\frac{2DV''_{\rm eff}(c)}{\pi}\right]^{1/2}\exp\left[\frac{-V_{\rm eff}(c)}{D}\right]\left[\int_{-\delta}^{+\delta}dy\exp\left[\frac{V_{\rm eff}(y)}{D}\right]\right]_{\rm num}^{-1}. \tag{42}$$

To compute α_1 we interchange the order of integration and differentiation to obtain

$$\alpha_1=-2\frac{df}{d(\eta\cos(\omega t))}\bigg|_{\eta=0} \tag{43}$$

$$=\left[\frac{2DV''_{\rm eff}(c)}{\pi}\right]^{1/2}\exp\left[\frac{-V_{\rm eff}(c)}{D}\right]\left[\int_{-\delta}^{+\delta}dy\exp\left[\frac{V_{\rm eff}(y)}{D}\right]\right]_{\rm num}^{-1} \tag{44}$$

$$+\left[\frac{2DV''_{\rm eff}(c)}{\pi}\right]^{1/2}\exp\left[\frac{-V_{\rm eff}(c)}{D}\right]\left[\int_{-\delta}^{+\delta}dy\exp\left[\frac{V_{\rm eff}(y)}{D}\right]\right]_{\rm num}^{-2}\left[\int_{-\delta}^{+\delta}dy\frac{y}{c}\exp\left[\frac{V_{\rm eff}(y)}{D}\right]\right]_{\rm num} \tag{45}$$

$$=\alpha_0\left\{1+\left[\int_{-\delta}^{+\delta}dy\frac{y}{c}\exp\left[\frac{V_{\rm eff}(y)}{D}\right]\right]_{\rm num}\left[\int_{-\delta}^{+\delta}dy\exp\left[\frac{V_{\rm eff}(y)}{D}\right]\right]_{\rm num}^{-1}\right\} \tag{46}$$

$$=\alpha_0\,, \tag{47}$$

where the integral in the numerator of the penultimate equality vanishes due to antisymmetry of the intgrand and symmetry of the range of integration. We note that this same equality of α_0 and α_1 was obtained by McNamara and Weisenfeld in their analysis of the classical system [5].

These two expressions complete our analysis. We are now ready to compute the SNR numerically for various cases.

IV. RESULTS

Our results are presented in Figs. 6–9. In Fig. 6, we have plots of the *normalized* SNR for the ammonia molecule (as shown in Fig. 2). The parameter $\xi=148.8$ and the system was modulated with a driving amplitude of $\epsilon=0.25$. This corresponds to a modulation of the barrier height by approximately 10%. In Fig. 6, curve B shows the semiquantal result compared to curve A, which is the classical result. Both have the characteristic stochastic resonance peak. However, there are significant differences: (i) The semiquantal SNR has a magnitude of the peak about twice that of the classical system; (ii) the peak is also visibly narrower, i.e., it is a sharper resonance; (iii) most significantly, however, the value of the noise variance D at resonance for the classical system is

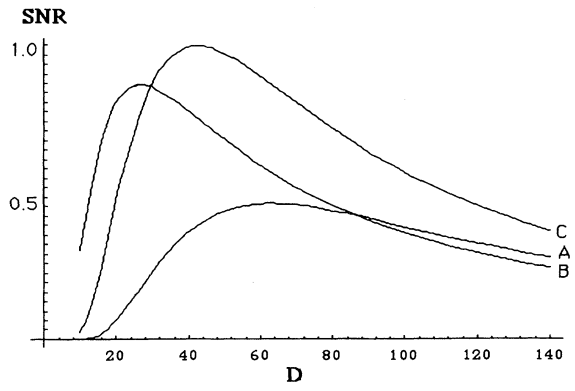

FIG. 6. Scaled SNR for $\xi=148.8$, $\epsilon=0.25$, A, classical; B, semiquantal; C, diminished classical.

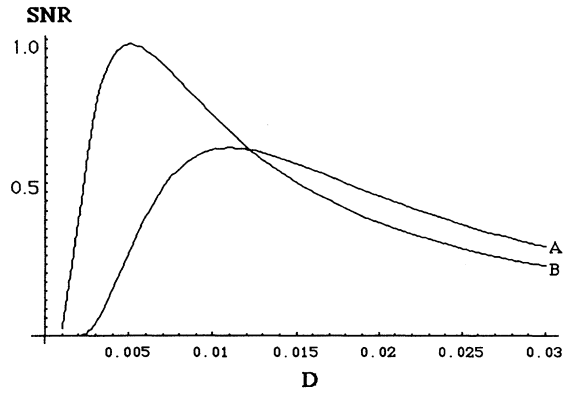

FIG. 7. Scaled SNR for $\xi=2.0$, $\epsilon=0.000\,25$. *A*, classical; *B*, semiquantal.

at D approximately equal to the barrier height, i.e., $D=60$. The peak for the semiquantal system is at $D=25$. This is both lower than the classical value and the approximate semiquantal barrier height $\delta V_{\text{eff}}=40$.

To verify that the semiquantal result is not merely due to the lower barrier height, we have plotted in Fig. 6, curve *C*, the SNR for the classical ammonia system but with barrier height artificially lowered to that of the semiquantal system. This system peaks, as expected, at $D=40$. The difference between the curves *C* and *A* demonstrates that the semiquantal geometry matters as well as the adjustment of barrier height.

In Fig. 7, we show the classical and semiquantal SNR plot for $\xi=2.0$. This normalized plot shows the same characteristics as the previous one. (The actual height of these peaks is, however, at $\sim 10^{-6}$.) As the dimensions of the system shrink, the difference between the classical and semiquantal barriers increases. This complicates our choice of driving amplitude, since one which is "small" for the classical system may not be so for the semiquantal system. In this figure, for example, we choose

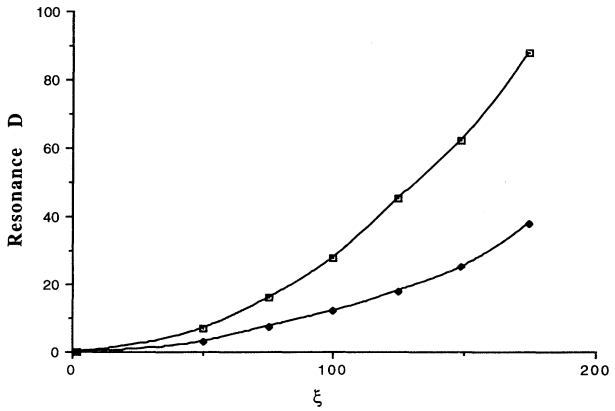

FIG. 8. Classical (upper curve) and semiquantal (lower curve) resonant values for D as a function of ξ. In each case ϵ equaled 10% of the semiquantal barrier height.

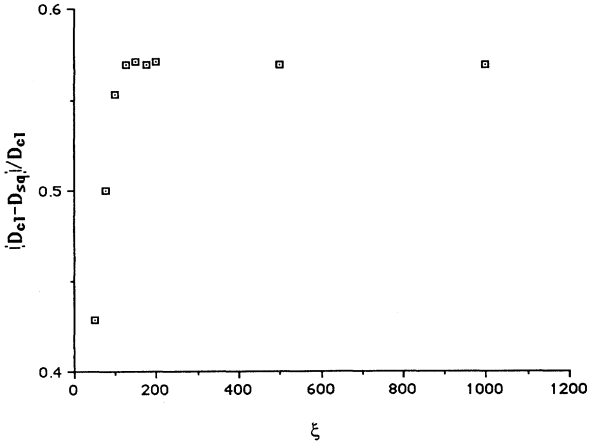

FIG. 9. Relative semiquantal shift in resonant value for D from the classical value as a function of ξ (ϵ as in Fig. 8).

$\epsilon=0.000\,25$ which is approximately 10% of the semiquantal barrier height but is about 5% of the classical height. The same essential effects as in the previous case are seen, irrespective.

We do not take our calculations to lower ξ; in this regime the presence of the central minimum invalidates the two-state master equation approach of McNamara and Weisenfeld. However, as argued before, in the limit $\xi\to 0$, the central minimum dominates the geometry of the system and implies the absence of stochastic resonance, in agreement with Löfstedt and Coppersmith [8].

We can, however, compute reasonable plots of the SNR for larger values of ξ. These are summarized in Figs. 8 and 9. In Fig. 8 we plot the value of D at which the SNR peaks for various ξ for the classical and semiquantal systems. The systems were driven at about 10% of the semiquantal barrier height for these calculations. In this figure we see the divergence between the classical and semiquantal results. Figure 9 shows the asymptotics that we expect to see as quantal effects become less significant at large scales. The relative semiquantal shift in the resonance value D is plotted versus ξ in this figure. This shift saturates at about 57%—its asymptotic existence is expected because the limit $\hbar\to 0$ (or in our $\xi\to\infty$) is singular in monitoring tunneling effects. There always exists a difference due to the inclusion of tunneling effects at any length scale.

We have thus estimated the general effect of including quantal effects on the classical phenomenon of stochastic resonance—it is a downward shift in the resonant value of the nose variance, an increase in the size of the resonance, and a sharpening of the resonance peak. At the extreme quantal limit, however, the phenomenon of stochastic resonance vanishes for the symmetric double-well system in agreement with recent calculations.

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